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ON SPECTRAL DENSITY ESTIMATES FOR A GAUSSIAN PERIODICALLY CORRELATED RANDOM FIELD

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Abstract. We consider a random field $\xi(t)$, $t = (t_1, t_2) \in R^2$, having mean value zero and the correlation function $B(t, \tau) = B(t_1, t_2, \tau_1, \tau_2) = E\xi(t_1+\tau_1, t_2+\tau_2)\xi(t_1, t_2)$, which is periodic in the sense that $B(t_1+T_1, t_2+T_2, \tau) \equiv B(t_1+T_1, t_2, \tau) \equiv B(t_1, t_2, \tau)$ (here the periods T_1 and T_2 are positive). It is shown that under broad conditions the spectral decomposition of the correlation function $B(t, \tau)$ is represented by the countable set of spectral densities $f_{j_1 j_2}(\lambda_1, \lambda_2)$, where $(j_1, j_2) \in Z^2$ and $(\lambda_1, \lambda_2) \in R^2$. For the case where the random field under consideration is Gaussian, nonparametric estimates of the spectral densities $f_{j_1 j_2}(\lambda_1, \lambda_2)$ are introduced and studied.

A random process $\xi(t)$, $t \in R$, is called *periodically correlated* (or *cyclostationary*) with period $T > 0$ if its mean value $m(t) = E\xi(t)$ and correlation function

$$B(t, \tau) = E\{[\xi(t+\tau)-m(t+\tau)][\xi(t)-m(t)]\}$$

are periodic functions of t with period $T > 0$. The periodically correlated random processes (and also slightly more general periodically nonstationary processes) are studied, e.g., in [1–4, 7], [5], Section 59, and [6], Section 26.5. Generalization of the concept of a periodically correlated random process to random functions of two variables leads to the concept of a periodically correlated random field. A random field $\xi(t)$, $t = (t_1, t_2) \in R^2$, is called *periodically correlated* with periods T_1 and T_2 if its mean value $m(t) = E\xi(t)$ and correlation function

$$\begin{aligned} B(t, \tau) &= B(t_1, t_2, \tau_1, \tau_2) \\ &= E\{[\xi(t_1, t_2)-m(t_1, t_2)][\xi(t_1+\tau_1, t_2+\tau_2)-m(t_1+\tau_1, t_2+\tau_2)]\} \end{aligned}$$

are periodic in the following sense:

$$m(t_1+T_1, t_2+T_2) \equiv m(t_1+T_1, t_2) \equiv m(t_1, t_2)$$

and

$$B(t_1+T_1, t_2+T_2, \tau) \equiv B(t_1+T_1, t_2, \tau) \equiv B(t_1, t_2, \tau).$$

It will be shown that the spectral decomposition of the correlation function of a periodically correlated random field can be described, as in the case of a periodically correlated random process, by the countable set of spectral densities. In this paper the most attention is given to the study of nonparametric estimates of the spectral densities for Gaussian periodically correlated random fields.

Let $\xi(t) = \xi(t_1, t_2)$ be a Gaussian real-valued periodically correlated random field, which has mean value zero and correlation function

$$B(t_1, t_2, \tau_1, \tau_2) = E\xi(t_1+\tau_1, t_2+\tau_2)\xi(t_1, t_2), \tag{1}$$

that is periodic in t_1 and t_2 with periods T_1 and T_2, respectively. In what follows we assume that for any $\tau = (\tau_1, \tau_2)$ the function $B(t, \tau)$ can be represented by its Fourier series

$$B(t_1, t_2, \tau) = \sum_{j_1 \in Z} \sum_{j_2 \in Z} B_{j_1 j_2}(\tau) e^{ij_1\omega_1 t_1} e^{ij_2\omega_2 t_2},$$

where $\omega_k = 2\pi/T_k$, $k = 1, 2$, and

$$B_{j_1 j_2}(\tau) = (T_1 T_2)^{-1} \int_0^{T_1} dt_1 \int_0^{T_2} \exp[-i(j_1\omega_1 t_1 + j_2\omega_2 t_2)] B(t_1, t_2, \tau) dt_2. \tag{2}$$

Assuming that the functions $|B_{j_1 j_2}(\tau_1, \tau_2)|$ decrease rapidly enough as $\tau_1^2 + \tau_2^2 \to \infty$, we obtain

$$B(t_1, t_2, \tau_1, \tau_2) = \sum_{j_1 \in Z} \sum_{j_2 \in Z} \exp[i(j_1\omega_1 t_1 + j_2\omega_2 t_2)] \times \int\int \exp[i(\lambda_1\tau_1 + \lambda_2\tau_2)] f_{j_1 j_2}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2, \tag{3}$$

where

$$\begin{aligned} f_{j_1 j_2}(\lambda_1, \lambda_2) &= (2\pi)^{-2} \int\int \exp[-i(\lambda_1\tau_1 + \lambda_2\tau_2)] B_{j_1 j_2}(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ &= (4\pi^2 T_1 T_2)^{-1} \int_0^{T_1} dt_1 \int_0^{T_2} \exp[-i(j_1\omega_1 t_1 + j_2\omega_2 t_2)] dt_2 \\ &\quad \times \int\int \exp[-i(\lambda_1\tau_1 + \lambda_2\tau_2)] B(t_1, t_2, \tau_1, \tau_2) d\tau_1 d\tau_2. \end{aligned} \tag{4}$$

Here (and in the sequel) the integral without integration limits denotes the integration from $-\infty$ to $+\infty$.

The spectral densities $f_{j_1 j_2}(\lambda)$, $\lambda = (\lambda_1, \lambda_2)$, defined by (4), are generally complex and satisfy the conditions

$$f_{j_1 j_2}(\lambda_1, \lambda_2) = \overline{f_{-j_1, -j_2}(-\lambda_1, -\lambda_2)} = f_{j_1 j_2}(j_1\omega_1 - \lambda_1, j_2\omega_2 - \lambda_2). \tag{5}$$

However, it follows from (5) that the density $f_{00}(\lambda)$ is always real. Let us show that $f_{00}(\lambda)$ is also a nonnegative function, and therefore it has all the properties of the spectral density function of a homogeneous random field in the plane. By (4), for proving the nonnegativity of $f_{00}(\lambda)$, it is sufficient to show that the function $B_{00}(\tau)$ is nonnegative definite, i.e. that the following two relations are valid:

(i) $B_{00}(-\tau_1, -\tau_2) = B_{00}(\tau_1, \tau_2)$;

(ii) for any $n \in N$, $\tau_j = (\tau_{j1}, \tau_{j2}) \in R^2$ and $c_j \in R$, $j = 1, \ldots, n$,

$$\sum_{k=1}^{n} \sum_{l=1}^{n} c_k c_l B_{00}(\tau_k - \tau_l) \geqslant 0.$$

The relation (i) follows easily from the definitions of $B(t, \tau)$ and $B_{00}(\tau)$. Moreover, using definitions (1) and (2) and the periodicity property of $B(t, \tau)$, we obtain

$$\sum_{k=1}^{n} \sum_{l=1}^{n} c_k c_l B_{00}(\tau_k - \tau_l) = \sum_{k=1}^{n} \sum_{l=1}^{n} c_k c_l B_{00}(\tau_{k1} - \tau_{l1}, \tau_{k2} - \tau_{l2})$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{c_k c_l}{T_1 T_2} \int_0^{T_1} dt_1 \int_0^{T_2} B(t_1, t_2, \tau_{k1} - \tau_{l1}, \tau_{k2} - \tau_{l2}) dt_2$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} c_k c_l (T_1 T_2)^{-1} \int_0^{T_1} dt_1 \int_0^{T_2} B(t_1 + \tau_{l1}, t_2 + \tau_{l2}, \tau_{k1} - \tau_{l1}, \tau_{k2} - \tau_{l2}) dt_2$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} c_k c_l (T_1 T_2)^{-1} \int_0^{T_1} dt_1 \int_0^{T_2} E[\xi(t_1 + \tau_{l1}, t_2 + \tau_{l2}) \xi(t_1 + \tau_{k1}, t_2 + \tau_{k2})] dt_2$$

$$= (T_1 T_2)^{-1} \int_0^{T_1} dt_1 \int_0^{T_2} E\Big|\sum_{k=1}^{n} c_k \xi(t_1 + \tau_{k1}, t_2 + \tau_{k2})\Big|^2 dt_2 \geqslant 0.$$

Hence the relation (ii) is also valid.

In the study of statistical estimation of the spectral densities $f_{j_1 j_2}(\lambda)$ it seems to be natural to assume that the random field $\xi(t)$ is harmonizable in the sense of [2], Section 4, and, consequently,

$$\sum_{j_1 \in Z} \sum_{j_2 \in Z} \iint |f_{j_1 j_2}(\lambda_1, \lambda_2)| d\lambda_1 d\lambda_2 < \infty. \tag{6}$$

However, it will be more convenient, instead of (6) to take the following assumption:

$$\sup_{\lambda} |f_{j_1 j_2}(\lambda)| \leqslant K_{j_1 j_2}, \qquad \sum_{j_1 \in Z} \sum_{j_2 \in Z} K_{j_1 j_2} = K < \infty. \tag{7}$$

Conditions (7) are clearly not too restrictive. It follows from the Fourier series theory that they are fulfilled if:

(i) for any $t \in R^2$, $\int\int |B(t, \tau_1, \tau_2)| d\tau_1 d\tau_2 < 4\pi^2 K_{00}$;

(ii) for any $\lambda \in R^2$ and $\Delta s \in R$ the functions

$$g_1(s; \lambda_1, \lambda_2) = (4\pi^2 T_2)^{-1} \frac{\partial}{\partial s} \Big[\int\int e^{-i(\lambda_1\tau_1+\lambda_2\tau_2)} d\tau_1 d\tau_2 \int_0^{T_2} B(s, t, \tau_1, \tau_2) dt\Big]$$

and

$$g_2(s; \lambda_1, \lambda_2) = (4\pi^2 T_1)^{-1} \frac{\partial}{\partial s} \Big[\int\int e^{-i(\lambda_1\tau_1+\lambda_2\tau_2)} d\tau_1 d\tau_2 \int_0^{T_1} B(t, s, \tau_1, \tau_2) dt\Big]$$

satisfy the condition

$$|g_j(s+\Delta s; \lambda) - g_j(s; \lambda)| \leqslant C_1 |\Delta s|^{\alpha_1}, \quad j = 1, 2,$$

where $\alpha_1 \in (0, 1]$ and $C_1 < \infty$;

(iii) for any $\lambda \in R^2$ and $(\Delta t_1, \Delta t_2) \in R^2$ the finite differences

$$A(t_1, t_2, \Delta t_1, \Delta t_2) = g(t_1+\Delta t_1, t_2+\Delta t_2) - g(t_1+\Delta t_1, t_2) - g(t_1, t_2+\Delta t_2) + g(t_1, t_2)$$

of the function

$$g(t_1, t_2) = g(t_1, t_2; \lambda_1, \lambda_2) = (2\pi)^{-2} \frac{\partial^2}{\partial t_1 \partial t_2} \Big[\int\int e^{-i(\lambda_1\tau_1+\lambda_2\tau_2)} B(t_1, t_2, \tau_1, \tau_2) d\tau_1 d\tau_2\Big]$$

satisfy the condition

$$|A(t_1, t_2, \Delta t_1, \Delta t_2)| \leqslant C_2 |\Delta t_1 \Delta t_2|^{\alpha_2},$$

where $\alpha_2 \in (0, 1]$ and $C_2 < \infty$.

Henceforth we denote the random field under consideration and its observed realization by the same symbol $\xi(t)$. As an estimate of $f_{k_1k_2}(\lambda)$, where $(k_1, k_2) \in Z^2$ and $\lambda \in R^2$, we consider the random variable

$$\begin{aligned}(8)\quad & f_{k_1k_2}^{(N_1,N_2)}(\lambda_1, \lambda_2) \\ &= (8\pi^2)^{-1} \int\int \big[\exp(-i(\lambda_1\tau_1+\lambda_2\tau_2)) + \exp(-i(k_1\omega_1-\lambda_1)\tau_1 - i(k_2\omega_2-\lambda_2)\tau_2)\big] \\ &\quad \times W(h\tau_1) W(h\tau_2) B_{k_1k_2}^{(N_1,N_2)}(\tau_1, \tau_2) d\tau_1 d\tau_2 \\ &= \frac{1}{4\pi^2} \int\int \exp\Big(-i\frac{k_1\omega_1\tau_1+k_2\omega_2\tau_2}{2}\Big) W(h\tau_1) W(h\tau_2) \\ &\quad \times \cos\frac{(k_1\omega_1-2\lambda_1)\tau_1+(k_2\omega_2-2\lambda_2)\tau_2}{2} B_{k_1k_2}^{(N_1,N_2)}(\tau_1, \tau_2) d\tau_1 d\tau_2,\end{aligned}$$

where

$$(9)\quad B_{k_1k_2}^{(N_1,N_2)}(\tau_1, \tau_2) = (T_1T_2)^{-1}\int_0^{T_1} dt_1 \int_0^{T_2} \exp[-i(k_1\omega_1 t_1 + k_2\omega_2 t_2)]B^{(N_1,N_2)}(t_1, t_2, \tau_1, \tau_2)dt_2,$$

$$(10)\quad B^{(N_1,N_2)}(t_1, t_2, \tau_1, \tau_2) = \sum_{l_1=-N_1}^{N_1}\sum_{l_2=-N_2}^{N_2}(n_1n_2)^{-1}\xi(t_1+l_1T_1+\tau_1, t_2+l_2T_2+\tau_2)\xi(t_1+l_1T_1, t_2+l_2T_2),$$

$$(11)\qquad n_k = 2N_k+1, \quad k = 1, 2,$$

$h = h(N_1, N_2)$ is a sequence of positive numbers such that $h+(h^2N_1N_2)^{-1}\to 0$ as $N_1N_2\to\infty$, and $W(x)$, $x\in R$, is a weighting function satisfying the conditions

$$(12)\qquad W(0) = 1, \quad W(-x) = W(x),$$

$$(13)\qquad W(x) = 0 \quad \text{if } |x|\geqslant 1.$$

An even integer $r\geqslant 2$ will be called the *order of the weighting function* $W(x)$ (or the *order of the statistical estimate* (8)) if $W(x)$ satisfies the following smoothness conditions:

(i) the derivative $W^{(r+1)}(x)$ exists for all $x\in R$ and is square integrable;
(ii) $W^{(l)}(0) = 0$, $l = 1, \ldots, r-1$, $W^{(r)}(0)\neq 0$.

We shall not consider the weighting functions $W(x)$ which do not satisfy the condition $\int[W^{(3)}(x)]^2dx<\infty$.

By the convolution theorem for the Fourier transforms, the estimate (8) can be also represented in the form

$$(14)\quad f_{k_1k_2}^{(N_1,N_2)}(\lambda_1, \lambda_2) = (2h^2)^{-1}\int\int I_{k_1k_2}^{(N_1,N_2)}(\mu_1, \mu_2)\left[w\left(\frac{\mu_1-\lambda_1}{h}\right)w\left(\frac{\mu_2-\lambda_2}{h}\right)+ +w\left(\frac{\mu_1-k_1\omega_1+\lambda_1}{h}\right)w\left(\frac{\mu_2-k_2\omega_2+\lambda_2}{h}\right)\right]d\mu_1d\mu_2,$$

where

$$(15)\quad I_{k_1k_2}^{(N_1,N_2)}(\lambda_1, \lambda_2) = (2\pi)^{-2}\int\int e^{-i(\lambda_1\tau_1+\lambda_2\tau_2)}B_{k_1k_2}^{(N_1,N_2)}(\tau_1, \tau_2)d\tau_1d\tau_2,$$

$$w(\lambda) = (2\pi)^{-1}\int_{-1}^{1} e^{-i\lambda x}W(x)dx.$$

Moreover, conditions (12) imply that

$$(16)\qquad w(-\lambda) = w(\lambda), \quad \int w(\lambda)d\lambda = 1.$$

Supplementary conditions (i) and (ii), imposed on the r-th order weighting function $W(x)$, lead to the formulae

$$(17)\qquad \int\lambda^{2r+2}w^2(\lambda)d\lambda<\infty$$

and

$$(18)\qquad \int \lambda^l w(\lambda)d\lambda = 0, \quad l = \overline{1,\ r-1}, \quad \int \lambda^r w(\lambda)d\lambda \neq 0,$$

respectively. Note also that the continuity of the r-th order weighting function $W(x)$ and the assumption (13) imply that

$$(19)\qquad \int w^2(\lambda)d\lambda = (2\pi)^{-1}\int W^2(x)dx < \infty.$$

A rather simple sequence of the r-th order weighting functions $W_r(x)$, $r = 2, 4, \ldots$, can be described by the formula

$$W_r(x) = \begin{cases}(1-x^r)^{r+2}, & |x| \leqslant 1, \\ 0, & |x| \geqslant 1.\end{cases}$$

It is clear that the estimate (8) of the spectral density $f_{k_1k_2}(\lambda)$ depends only on the values of the realization $\xi(t)$ in the rectangle

$$\Delta = \{(t_1, t_2):\ t_1 \in (-N_1 T_1 - 1/h,\ (N_1+1)T_1 + 1/h),$$
$$t_2 \in (-N_2 T_2 - 1/h,\ (N_2+1)T_2 + 1/h)\},$$

which is expanding infinitely (along both coordinate axes) as $N_1 N_2 \to \infty$.

THEOREM. *Let the spectral densities $f_{j_1j_2}(\lambda)$, $(j_1, j_2) \in Z^2$, satisfy conditions* (7) *and assume that the real and imaginary parts of the spectral density $f_{k_1k_2}(\lambda)$, which is to be estimated, have the bounded partial derivatives of all the orders we are interested in. Then the r-th order estimate* (8) *of the spectral density $f_{k_1k_2}(\lambda)$ satisfies the asymptotical relation*

$$E|f_{k_1k_2}^{(N_1,N_2)}(\lambda) - f_{k_1k_2}(\lambda)|^2 = O\left((N_1 N_2)^{-r/(r+1)}\right)$$

if $h = h(N_1, N_2)$ is chosen to be proportional to $(N_1 N_2)^{-1/(2r+2)}$ as $N_1 N_2 \to \infty$.

Proof. It is easy to see that equations (15), (9), (10), (1) and (4) imply that $EI_{k_1k_2}^{(N_1,N_2)}(\lambda) = f_{k_1k_2}(\lambda)$. Therefore it follows from (14), (5) and (16) that

$$b[f_{k_1k_2}^{(N_1,N_2)}(\lambda_1, \lambda_2)] \stackrel{\text{def}}{=} Ef_{k_1k_2}^{(N_1,N_2)}(\lambda_1, \lambda_2) - f_{k_1k_2}(\lambda_1, \lambda_2)$$

$$= \frac{1}{2h^2}\int\int f_{k_1k_2}(\mu_1, \mu_2)\left[w\left(\frac{\mu_1 - k_1\omega_1 + \lambda_1}{h}\right)w\left(\frac{\mu_2 - k_2\omega_2 + \lambda_2}{h}\right)\right.$$

$$\left. + w\left(\frac{\mu_1 - \lambda_1}{h}\right)w\left(\frac{\mu_2 - \lambda_2}{h}\right)\right]d\mu_1 d\mu_2 - f_{k_1k_2}(\lambda_1, \lambda_2)$$

$$= h^{-2}\int\int w\left(\frac{\mu_1 - \lambda_1}{h}\right)w\left(\frac{\mu_2 - \lambda_2}{h}\right)[f_{k_1k_2}(\mu_1, \mu_2) - f_{k_1k_2}(\lambda_1, \lambda_2)]d\mu_1 d\mu_2.$$

Hence, expanding $f_{k_1k_2}(\mu_1, \mu_2)$ into the Taylor series in the neighbourhood of the point (λ_1, λ_2) and taking into account assumptions (18), we obtain

$$(20) \qquad |b[f_{k_1k_2}^{(N_1,N_2)}(\lambda_1, \lambda_2)]| \leqslant \frac{2M_r h^r}{r!} \int\int |w(\mu_1)w(\mu_2)|(|\mu_1|+|\mu_2|)^r d\mu_1 d\mu_2,$$

where M_r is the upper bound of the absolute values of all the r-th order partial derivatives of the real and imaginary parts of the spectral density $f_{k_1k_2}(\lambda_1, \lambda_2)$.

It is now easy to show that

$$(21) \qquad \int\int |w(\mu_1)w(\mu_2)|(|\mu_1|+|\mu_2|)^r d\mu_1 d\mu_2 < \infty$$

if

$$(22) \qquad \int |\lambda^l w(\lambda)| d\lambda < \infty, \qquad l = 0, 1, \ldots, r.$$

For $\lambda \in R$ let $\varphi(\lambda) = \min[1, |\lambda|^{-1}]$ and $\psi(\lambda) = \max[1, |\lambda|^{r+1}]$. By Cauchy's inequality we find that for any $l = 0, 1, \ldots, r$

$$(23) \qquad [\int |\lambda^l w(\lambda)| d\lambda]^2 \leqslant [\int \varphi(\lambda)\psi(\lambda)|w(\lambda)| d\lambda]^2$$

$$\leqslant \int \psi^2(\lambda)w^2(\lambda)d\lambda \int \varphi^2(\mu)d\mu = 4\int \psi^2(\lambda)w^2(\lambda)d\lambda \leqslant 4\int (1+\lambda^{2r+2})w^2(\lambda)d\lambda.$$

Moreover (23), (17) and (19) imply (22) and (21).

Let us now consider the variance of the estimate (8). Clearly,

$$(24) \qquad \mathrm{Var}[f_{k_1k_2}^{(N_1,N_2)}(\lambda_1, \lambda_2)] = E|f_{k_1k_2}^{(N_1,N_2)}(\lambda_1, \lambda_2)|^2 - |Ef_{k_1k_2}^{(N_1,N_2)}(\lambda_1, \lambda_2)|^2$$

$$= \frac{1}{16\pi^4 T_1^2 T_2^2} \int\int \exp\left(-i\frac{k_1\omega_1\tau_1 + k_2\omega_2\tau_2}{2}\right) W(h\tau_1)$$

$$\times W(h\tau_2)\cos\frac{(k_1\omega_1 - 2\lambda_1)\tau_1 + (k_2\omega_2 - 2\lambda_2)\tau_2}{2} d\tau_1 d\tau_2$$

$$\times \int\int W(h\tau_3)W(h\tau_4)\cos\frac{(k_1\omega_1 - 2\lambda_1)\tau_3 + (k_2\omega_2 - 2\lambda_2)\tau_4}{2}$$

$$\times \exp\left(i\frac{k_1\omega_1\tau_3 + k_2\omega_2\tau_4}{2}\right) d\tau_3 d\tau_4 \int_0^{T_1} e^{-ik_1\omega_1 t_1} dt_1$$

$$\times \int_0^{T_2} e^{-ik_2\omega_2 t_2} dt_2 \int_0^{T_1} e^{ik_1\omega_1 t_3} dt_3 \int_0^{T_2} e^{ik_2\omega_2 t_4}$$

$$\times \{E[B^{(N_1,N_2)}(t_1, t_2, \tau_1, \tau_2)B^{(N_1,N_2)}(t_3, t_4, \tau_3, \tau_4)]$$

$$- EB^{(N_1,N_2)}(t_1, t_2, \tau_1, \tau_2)EB^{(N_1,N_2)}(t_3, t_4, \tau_3, \tau_4)\} dt_4.$$

By definition (10) and the Gaussianity assumption, we also obtain

$$E[B^{(N_1,N_2)}(t_1, t_2, \tau_1, \tau_2)B^{(N_1,N_2)}(t_3, t_4, \tau_3, \tau_4)] - EB^{(N_1,N_2)}(t_1, t_2, \tau_1, \tau_2)EB^{(N_1,N_2)}(t_3, t_4, \tau_3, \tau_4)$$

$$= \sum_{l_1=-N_1}^{N_1} \sum_{l_2=-N_2}^{N_2} \sum_{l_3=-N_1}^{N_1} \sum_{l_4=-N_2}^{N_2} (n_1 n_2)^{-2} \times \{E[\xi(t_1+l_1 T_1, t_2+l_2 T_2)\xi(t_1+l_1 T_1+\tau_1, t_2+l_2 T_2+\tau_2)$$
$$\times \xi(t_3+l_3 T_1, t_4+l_4 T_2)\xi(t_3+l_3 T_1+\tau_3, t_4+l_4 T_2+\tau_4)]$$
$$-E\xi(t_1+l_1 T_1, t_2+l_2 T_2)\xi(t_1+l_1 T_1+\tau_1, t_2+l_2 T_2+\tau_2)$$
$$\times E\xi(t_3+l_3 T_1, t_4+l_4 T_2)\xi(t_3+l_3 T_1+\tau_3, t_4+l_4 T_2+\tau_4)\}$$

$$= \sum_{l_1=-N_1}^{N_1} \sum_{l_2=-N_2}^{N_2} \sum_{l_3=-N_1}^{N_1} \sum_{l_4=-N_2}^{N_2} (n_1 n_2)^{-2} \times \big[B\big(t_3, t_4, t_1-t_3+(l_1-l_3)T_1, t_2-t_4+(l_2-l_4)T_2\big)$$
$$\times B\big(t_1+\tau_1, t_2+\tau_2, t_3-t_1+(l_3-l_1)T_1+\tau_3-\tau_1, t_4-t_2+(l_4-l_2)T_2+\tau_4-\tau_2\big)$$
$$+B\big(t_3, t_4, t_1-t_3+(l_1-l_3)T_1+\tau_1, t_2-t_4+(l_2-l_4)T_2+\tau_2\big)$$
$$\times B\big(t_1, t_2, t_3-t_1+(l_3-l_1)T_1+\tau_3, t_4-t_2+(l_4-l_2)T_2+\tau_4\big)\big].$$

Using now (3) and performing some rather simple transforms, we find that

$$(25)\quad E[B^{(N_1,N_2)}(t_1, t_2, \tau_1, \tau_2)B^{(N_1,N_2)}(t_3, t_4, \tau_3, \tau_4)] - EB^{(N_1,N_2)}(t_1, t_2, \tau_1, \tau_2)EB^{(N_1,N_2)}(t_3, t_4, \tau_3, \tau_4)$$
$$= \sum_{j_1\in Z} e^{ij_1\omega_1 t_1} \sum_{j_2\in Z} e^{ij_2\omega_2 t_2} \sum_{j_3\in Z} e^{ij_3\omega_1 t_3} \sum_{j_4\in Z} e^{ij_4\omega_2 t_4} \iint e^{i\mu_1(t_3-t_1+\tau_3)} e^{i\mu_2(t_4-t_2+\tau_4)}$$
$$\times f_{j_1 j_2}(\mu_1, \mu_2)d\mu_1 d\mu_2 \iint \frac{\sin^2[n_1 T_1(\mu_3-\mu_1)/2]}{n_1^2\sin^2[T_1(\mu_3-\mu_1)/2]} \frac{\sin^2[n_2 T_2(\mu_4-\mu_2)/2]}{n_2^2\sin^2[T_2(\mu_4-\mu_2)/2]}$$
$$\times [e^{i(j_1\omega_1-\mu_1)\tau_1} e^{i(j_2\omega_2-\mu_2)\tau_2} + e^{i(\mu_3\tau_1+\mu_4\tau_2)}] e^{i\mu_3(t_1-t_3)} e^{i\mu_4(t_2-t_4)} f_{j_3 j_4}(\mu_3, \mu_4)d\mu_3 d\mu_4.$$

Comparing now (24) with (25) we can write the variance of the estimate (8) in the form

$$\mathrm{Var}[f_{k_1 k_2}^{(N_1,N_2)}(\lambda)] = V_1(N_1, N_2)+V_2(N_1, N_2),$$

where $V_l(N_1, N_2)$, $l=1, 2$, is the summand produced by the l-th summand term in the last brackets entering the right-hand side of (25). In particular,

$$V_1(N_1, N_2) = \frac{1}{16\pi^4}\iint \cos\frac{(k_1\omega_1-2\lambda_1)\tau_1+(k_2\omega_2-2\lambda_2)\tau_2}{2}$$
$$\times W(h\tau_1)W(h\tau_2)\exp[-i(k_1\omega_1\tau_1+k_2\omega_2\tau_2)/2]d\tau_1 d\tau_2$$

$$\times \iint \cos\frac{(k_1\omega_1-2\lambda_1)\tau_3+(k_2\omega_2-2\lambda_2)\tau_4}{2} W(h\tau_3)W(h\tau_4)$$

$$\times \exp[i(k_1\omega_1\tau_3+k_2\omega_2\tau_4)/2]\,d\tau_3\,d\tau_4 \int_0^{T_1} e^{-ik_1\omega_1 t_1}dt_1$$

$$\times \int_0^{T_2} e^{-ik_2\omega_2 t_2}dt_2 \int_0^{T_1} e^{ik_1\omega_1 t_3}dt_3 \int_0^{T_2} e^{ik_2\omega_2 t_4}$$

$$\times \Big\{ \sum_{j_1\in Z} e^{ij_1\omega_1 t_1} \sum_{j_2\in Z} e^{ij_2\omega_2 t_2} \sum_{j_3\in Z} e^{ij_3\omega_1 t_3} \sum_{j_4\in Z} e^{ij_4\omega_2 t_4} \iint f_{j_1 j_2}(\mu_1,\mu_2) e^{i\mu_1(t_3-t_1+\tau_3-\tau_1)}$$

$$\times e^{i\mu_2(t_4-t_2+\tau_4-\tau_2)} e^{i(j_1\omega_1\tau_1+j_2\omega_2\tau_2)}\,d\mu_1\,d\mu_2$$

$$\times \iint \frac{\sin^2[n_1T_1(\mu_3-\mu_1)/2]}{n_1^2T_1^2\sin^2[T_1(\mu_3-\mu_1)/2]}\,\frac{\sin^2[n_2T_2(\mu_4-\mu_2)/2]}{n_2^2T_2^2\sin^2[T_2(\mu_4-\mu_2)/2]}$$

$$\times f_{j_3 j_4}(\mu_3,\mu_4) e^{i\mu_3(t_1-t_3)} e^{i\mu_4(t_2-t_4)}\,d\mu_3\,d\mu_4 \Big\}\,dt_4.$$

Integrating with respect to t_l $(l = 1, 2, 3, 4)$, we obtain

$$V_1(N_1, N_2) = \frac{1}{16\pi^4} \sum_{j_1\in Z}\sum_{j_2\in Z}\sum_{j_3\in Z}\sum_{j_4\in Z} (-1)^{j_1+j_2}$$

$$\times (-1)^{j_3+j_4} \iint f_{j_3 j_4}(\mu_3,\mu_4)\,d\mu_3\,d\mu_4 \iint f_{j_1 j_2}(\mu_1,\mu_2)$$

$$\times \Big\{ \prod_{\alpha=1}^{2} \frac{\sin[n_\alpha T_\alpha(\omega_\alpha(k_\alpha-j_\alpha)+\mu_\alpha-\mu_{\alpha+2})/2]}{n_\alpha T_\alpha(\omega_\alpha(k_\alpha-j_\alpha)+\mu_\alpha-\mu_{\alpha+2})/2}$$

$$\times \frac{\sin[n_\alpha T_\alpha(\omega_\alpha(k_\alpha+j_{\alpha+2})+\mu_\alpha-\mu_{\alpha+2})/2]}{n_\alpha T_\alpha(\omega_\alpha(k_\alpha+j_{\alpha+2})+\mu_\alpha-\mu_{\alpha+2})/2} \Big\} \varphi_{j_1 j_2}(\mu_1,\mu_2)\psi(\mu_1,\mu_2)\,d\mu_1\,d\mu_2,$$

where

$$(26)\qquad \varphi_{j_1 j_2}(\mu_1,\mu_2) = \iint \exp[-i(\mu_1\tau_1+\mu_2\tau_2)]\,W(h\tau_1)$$

$$\times W(h\tau_2)\exp[i\omega_1\tau_1(j_1-k_1/2)+i\omega_2\tau_2(j_2-k_2/2)]$$

$$\times \cos[(k_1\omega_1/2-\lambda_1)\tau_1+(k_2\omega_2/2-\lambda_2)\tau_2]\,d\tau_1\,d\tau_2,$$

$$(27)\qquad \psi(\mu_1,\mu_2) = \iint \exp[i(\mu_1\tau_3+\mu_2\tau_4)]\,W(h\tau_3)W(h\tau_4)$$

$$\times \exp[i(k_1\omega_1\tau_3+k_2\omega_2\tau_4)/2]\cos[(k_1\omega_1/2-\lambda_1)\tau_3+(k_2\omega_2/2-\lambda_2)\tau_4]\,d\tau_3\,d\tau_4.$$

Now, using the assumption (7) and Cauchy's inequality, we find that

$$|V_1(N_1, N_2)| \leqslant \sum_{j_1\in Z} \sum_{j_2\in Z} \sum_{j_3\in Z} \sum_{j_4\in Z} (4\pi^2 n_1 n_2)^{-1}$$

$$\times K_{j_1j_2} K_{j_3j_4} \Big\{ \int\int |\psi(\mu_1, \mu_2)|^2 d\mu_1 d\mu_2$$

$$\times \prod_{\alpha=1}^{2} \int \frac{1}{2\pi n_\alpha} \frac{\sin^2[n_\alpha T_\alpha(\omega_\alpha(k_\alpha - j_\alpha) + \mu_\alpha - \mu_{\alpha+2})/2]}{[T_\alpha(\omega_\alpha(k_\alpha - j_\alpha) + \mu_\alpha - \mu_{\alpha+2})/2]^2} d\mu_{\alpha+2} \Big\}^{1/2}$$

$$\times \Big\{ \int\int |\varphi_{j_1j_2}(\mu_1, \mu_2)|^2 d\mu_1 d\mu_2$$

$$\times \prod_{\alpha=1}^{2} \int \frac{1}{2\pi n_\alpha} \frac{\sin^2[n_\alpha T_\alpha(\omega_\alpha(k_\alpha + j_{\alpha+2}) + \mu_\alpha - \mu_{\alpha+2})/2]}{[T_\alpha(\omega_\alpha(k_\alpha + j_{\alpha+2}) + \mu_\alpha - \mu_{\alpha+2})/2]^2} d\mu_{\alpha+2} \Big\}^{1/2}$$

$$= \sum_{j_1\in Z} \sum_{j_2\in Z} \sum_{j_3\in Z} \sum_{j_4\in Z} \frac{K_{j_1j_2} K_{j_3j_4}}{4\pi^2 n_1 n_2 T_1 T_2}$$

$$\times \Big[\int\int |\varphi_{j_1j_2}(\mu_1, \mu_2)|^2 d\mu_1 d\mu_2 \int\int |\psi(\nu_1, \nu_2)|^2 d\nu_1 d\nu_2 \Big]^{1/2}.$$

Taking into account definitions (26) and (27) of $\varphi_{j_1j_2}(\mu_1, \mu_2)$ and $\psi(\mu_1, \mu_2)$, we get

$$(28) \quad |V_1(N_1, N_2)| \leqslant \frac{K^2}{4\pi^2 n_1 n_2 T_1 T_2} \int\int |\psi(\nu_1, \nu_2)|^2 d\nu_1 d\nu_2$$

$$= \frac{K^2}{n_1 n_2 T_1 T_2} \int\int \cos^2 \frac{(k_1\omega_1 - 2\lambda_1)\tau_1 + (k_2\omega_2 - 2\lambda_2)\tau_2}{2} W^2(h\tau_1) W^2(h\tau_2) d\tau_1 d\tau_2$$

$$= \frac{K^2}{n_1 n_2 h^2 T_1 T_2} \int_{-1}^{1} W^2(\tau_1) d\tau_1$$

$$\times \int_{-1}^{1} W^2(\tau_2) \cos^2 \frac{(k_1\omega_1 - 2\lambda_1)\tau_1 + (k_2\omega_2 - 2\lambda_2)\tau_2}{2h} d\tau_2.$$

It can be similarly shown that $|V_2(N_1, N_2)|$ also does not exceed the right-hand side of (28). Therefore

$$(29) \quad \mathrm{Var}[f_{k_1k_2}^{(N_1,N_2)}(\lambda)] \leqslant |V_1(N_1, N_2)| + |V_2(N_1, N_2)|$$

$$\leqslant \frac{2K^2}{n_1 n_2 h^2 T_1 T_2} \int_{-1}^{1} W^2(\tau_1) d\tau_1$$

$$\times \int_{-1}^{1} W^2(\tau_2) \cos^2 \{[(k_1\omega_1 - 2\lambda_1)\tau_1 + (k_2\omega_2 - 2\lambda_2)\tau_2]/2h\} d\tau_2.$$

Since $h = h(N_1, N_2) \to 0$ as $N_1 N_2 \to \infty$, and, by assumption, the function $W(\tau)$ is continuous, it follows from (29) that

$$(30) \quad \limsup_{N_1N_2\to\infty} n_1 n_2 h^2 \mathrm{Var}[f_{k_1k_2}^{(N_1,N_2)}(\lambda)]$$

$$\leqslant \frac{K^2}{T_1 T_2} \{2 - \mathrm{sign}[\sum_{j=1}^{2} (2\lambda_j - k_j\omega_j)^2]\} (\int_{-1}^{1} W^2(\tau) d\tau)^2.$$

Now the theorem follows from the formula

$$E|f_{k_1k_2}^{(N_1,N_2)}(\lambda)-f_{k_1k_2}(\lambda)|^2 = \mathrm{Var}[f_{k_1k_2}^{(N_1,N_2)}(\lambda)]+|b[f_{k_1k_2}^{(N_1,N_2)}(\lambda)]|^2$$

as well as from relations (20), (21), (30), (11) and the assumption that $h = h(N_1, N_2) \sim (N_1 N_2)^{-1/(2r+2)}$ as $N_1 N_2 \to \infty$.

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